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GALOIS THEORY FOR A CLASS OF COMPLETE MODULAR LATTICES

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Abstract. We construct Galois theory for sublattices of certain complete modular lattices and their automorphism groups. A well–known description of the intermediate subgroups of the general linear group over a semilocal ring containing the group of diagonal matrices, due to Z.I.Borewicz and N.A.Vavilov, can be obtained as a consequence of this theory. Bibliography: 3 titles.

Introduction

We generalize here the results of [PY], [S]. Namely, Galois theory for a class of complete modular lattices is constructed.

By an automorphism of a complete lattice we mean hereafter a bijective mapping of the lattice onto itself which commutes with the supremum and infimum of every subset of the lattice. Other notions and definitions are introduced in [PY].

Formulation of the main results

Let L be a complete modular lattice, L_0 its finite sublattice, which is a Boolean algebra, G a subgroup of the group of all automorphisms of the lattice L, $H = G(L_0)$.

Let $e_1, e_2, \dots, e_n$ be the atoms of L_0. We consider a number of additional conditions (it is supposed that, unless otherwise stated, the indices are changing from 1 to n):

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1^0. $0_{L_0} = 0_L,\ 1_{L_0} = 1_L$.

2^0. If $f \in G$ and $f(e_i) + \widehat{e}_j = 1$ for some i, j, then $f(e_i) \cdot \widehat{e}_j = 0$.

3^0. If $a \in G$ and $[a(e_i)]_i = e_i$ for some i, then there exists $h \in H_i$ such that $[ha(x_i)]_i = [ah(x_i)]_i = x_i$ for every $x_i \leqslant e_i$.

4^0. There exists $h \in H_t \cap G(\overline{L}_0)$ such that $[aha^{-1}(x_i)]_r = [a([a^{-1}(x_i)]_t)]_r$ for every $a \in G,\ r \neq i,\ x_i \leqslant e_i$.

5^0. Let $u \in \overline{L}_0,\ u \geqslant e_i$ for some i; let $g \in G,\ [g(u)]_i = e_i$. Then there exists $t \in G$ such that:

1) $[gt(e_i)]_i = e_i$,
2) $t(e_s) = e_s$ for $s \neq i$,
3) $[t(e_i)]_j \leqslant [u]_j$.

6^0. If $f, g \in G$ and $[f(e_i)]_j \leqslant [g(e_i)]_j$ for some i, j, then $[f(x)]_j \leqslant [g(x)]_j$ for every $x \in L_0^{'},\ x \leqslant e_i$.

7^0. If $u \leqslant e_j$ for some j, then for every $i \neq j$ there exist $y_\alpha \leqslant e_j, \alpha \in I$ such that $u = \sum\limits_{\alpha \in I} y_\alpha$ and $H_{ij}(y_\alpha) \neq \varnothing$.

8^0. If $x = [f(e_i)]_j$ for some $f \in G,\ i \neq j$, then there exists $g \in H_{ij}(x)$ such that $[g(u)]_j = [f(u)]_j$ for every $u \leqslant e_i$.

9^0. If $w \in L,\ [w] = (0, \dots, e_i, \dots, x, \dots, 0)$, where $w \cdot e_j = 0$ and $H_{ij}(x) \neq \varnothing$, then there exists $t \in H_{ij}(x)$ such that $t(w) = e_i$.

10^0. Let $a_\alpha \in H_{ij}(x_\alpha)$ and $y \leqslant \sum\limits_{\alpha \in I} x_\alpha$ be such that $H_{ij}(y) \neq \varnothing$. Then $H_{ij}(y) \subseteq \langle H, a_\alpha :\ \alpha \in I \rangle$.

11^0. If $a \in G$, then for every $i \neq j$ and every $h \in H_t$ the set $H_{ij}([aha^{-1}(e_i)]_j) \cap \langle a, H \rangle$ is not empty.

12^0. $L_0^{'} \subseteq \overline{L}_0$.

13^0. For every $x \leqslant e_i,\ x \neq e_i$ there exists a coatom $y \leqslant e_i$ such that $x \leqslant y$.

We denote $w_i = \prod\limits_{x\ is\ a\ coatom\ in\ e_i} x$ for every i and $w = \sum\limits_{i=1}^{n} w_i$.

14^0. The lattice $L^w = \{x \in L :\ x \geqslant w\}$ is of finite length.

We denote $G^w = \{g \in G :\ g$ is identical on $L^w\}$.

15^0. Let $t \in H_{ij}(x),\ x \leqslant w_j$. Then there exists $h \in H$ such that $th \in G^w$.

16^0. The lattice $L^w \cap \overline{L}_0(H)$ is finite.

Theorem 1. *Assuming that the conditions $1^0 - 12^0$ are fulfilled, for every subgroup $F \geqslant H$ of group G:*

(i) $\sigma = \sigma(F)$ is a net collection in $L_0^{'}$;

(ii) $G(K_\sigma) \trianglelefteq F$;

(iii) if M is a sublattice of $L_0^{'}$ such that $G(M) \trianglelefteq F$, then $G(M) = G(K_\sigma)$.

Theorem 2. *Let $\tau = (\tau_{ij})$ be a net collection in $L_0^{'}$, $g \in G$. Provided that the conditions $1^0 - 11^0,\ 13^0 - 16^0$ are fulfilled, we have:*

(i) if $[g(e_i)]_j \leqslant \tau_{ij}$ for every i, j, then $g \in G(K_\tau)$;

(ii) the index of $G(K_\tau)$ in its normalizer is finite.

Proof of the main results

Proof of the parts (i)–(iii) of Theorem 1 is analogous to the proof of the corresponding assertions of [PY]. We note that instead of properties of the dimension function on L used in [PY], one must apply the modularity law and the following trivial statement:

If $A \subseteq G$, where $A^{-1} = A$ and $a(x) \leqslant x$ for every $a \in A$ and some $x \in L$, then $a(x) = x$ for every $a \in A$.

Proof of Theorem 2 will be presented below.

Lemma 1. *For every $f \in G$ $f(w) = w$.*

Proof. It is sufficient to check $[f(w_i)]_j \leqslant w_j$ for every i, j.

It is clear that $w \in L_0^{'}$, therefore for $i = j$ it is just the condition 6^0.

Let $i \neq j,\ g \in H_{ij}(e_j)$. By the condition 9^0 there exists $t \in H_{ji}(e_i)$ such that $tg(e_i) = e_j$. Consider an arbitrary coatom y in e_j. Then $(tg)^{-1}(y)$ is a coatom in e_i, therefore $tg(w_i) \leqslant y$, whence $[g(w_i)]_j \leqslant w_j$. Now it remains to apply the condition 6^0.

Suppose $\tau = (\tau_{ij})$ is a net collection in $L_0^{'}$.

Lemma 2. *Let $\rho_{ij} = \tau_{ij} + w_j$. Then:*

(i) $\rho = (\rho_{ij})$ is a net collection in $L_0^{'}$;

(ii) $G(K_\rho) = G(K_\tau) \cdot G^w$.

Proof. Let i, j, k be pairwise distinct, and let $g \in H_{ij}(x),\ x \leqslant \tau_{ij} + w_j$.

By the conditions 7^0 and 10^0 $g \in \langle H, H_{ij}(y), H_{ij}(z) :\ y \leqslant \tau_{ij},\ z \leqslant w_j \rangle$.

If $f \in H_{ij}(y)$, where $y \leqslant \tau_{ij}$, then $[f(\tau_{ki} + w_i)]_j \leqslant \tau_{kj} + w_j$ by Lemma 1. If $f \in H_{ij}(z)$, where $z \leqslant w_j$, then $[f(\tau_{ki} + w_i)]_j \leqslant w_j$.

(i) Apply the condition 8^0.

(ii) The inclusion $\supseteq$ is trivial. Further, by the condition 15^0 and by Theorem 7.2 [PY] $G(K_\rho) \subseteq \langle G(K_\tau), G^w \rangle$. It remains to note that $G^w \trianglelefteq G$.

Proof of Theorem 2(i).

A. First, suppose that $\tau_{ij} \leqslant w_j$ for every $i \neq j$. Since $1 = \sum_{i=1}^{n} g(e_i)$, we have $e_1 = [g(e_1)]_1 + w_1$. By the condition 13^0 $[g(e_1)]_1 = e_1$. Repeating the proof of Theorem 7.2 [PY], we obtain $g \in \langle H,\ H_{ij}(x) :\ x \leqslant \tau_{ij} \rangle \leqslant G(K_\tau)$.

B. General case. We put $x_i = \sum_{j=1}^{n} \tau_{ij}$. By the definition of a net collection we have $g(x_i) \leqslant x_i$. Further, it follows from Lemma 1 that $g(x_i + w) \leqslant x_i + w$,

and since the restriction of g to L^w is an automorphism of this lattice, we have $g(x_i + w) = x_i + w$ by the condition 14^0. Thus $g \in G(K_\rho)$.

It follows from Lemma 2 that $g = g_1 g_2$, where $g_1 \in G(K_\tau)$, $g_2 \in G^w$. Since $g_2(x_i) \leqslant x_i$, then $[g_2(e_i)]_j \leqslant \tau_{ij} \cdot w_j$ for every $i \neq j$. We have already proved in the part A that $[g_2^{-1}(e_i)]_j \leqslant \tau_{ij} \cdot w_j$ for every $i \neq j$, therefore $g_2 \in G(K_\tau)$.

Proof of the part (ii) is conceptually identical with the constructions of § 7 of the article [BV].

As in [PY], a complete description of subgroups of the general linear group over a semilocal ring (whose fields of residues have at least seven elements, see [BV]), containing the group of diagonal matrices, can be deduced from Theorems 1 and 2.

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